1/10-5

Geometric Interpretation of the Function $g(c,\mu)$ of Neutron Transport Theory

Perry A. Newman
NASA Langley Research Center
Hampton, Virginia 23365

and

Robert L. Bowden
Department of Physics, Virginia Polytechnic Institute
Blacksburg, Virginia 24061

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In their classic monograph on neutron transport theory, Case et al.[1] introduced the function $g(c,\mu)$,

$$g(c,\mu) = [(1 - c\mu \tanh^{-1} \mu)^2 + (\pi c\mu/2)^2]^{-1} \quad , \qquad (1)$$

and gave many of its properties. It is now well known[2] that this function is simply related to the normalization of the singular eigenfunctions introduced later by Case.[3] An analogous function is readily defined[2] in time-dependent monoenergetic neutron-transport problems in plane geometry with isotropic scattering; however it also contains a complex transform variable. Bowden and Williams[4] have shown that in such time-dependent problems the discrete eigenfunctions are not always present in the normal-mode expansion of the Laplace transform of the solution; that is, there exists a curve, C_s, in the s/c-plane (see Fig. 1) such that if s/c lies inside the curve ($s \epsilon S_i$) discrete terms are present, whereas for s/c outside the curve ($s \epsilon S_e$) there are no discrete terms. The curve C_s is given by[4]

$$C_s = \left\{ s/c = \alpha' + i\beta' \,\middle|\, \alpha' = \frac{2\beta'}{\pi} \tanh^{-1} \frac{2\beta'}{\pi} \right\} \quad . \qquad (2)$$

We want to point out in this note that the function

$$\frac{1}{\alpha^2}\, g\left(\frac{1}{\alpha},\mu\right) = \left.\frac{c^2}{\Omega^+(\mu,s)\Omega^-(\mu,s)}\right|_{\beta=0} , \quad 0 \leq \mu \leq 1 \quad , \qquad (3)$$

where $s/c = \alpha + i\beta$ and $\Omega^{\pm}(\mu,s)$ are the limiting values of the dispersion function

$$\Omega(z,s) = s - cz \tanh^{-1} (1/z) \qquad (4)$$

on its branch cut in the z-plane, is the inverse of the square of the distance in the s/c-plane from the point $(\alpha,0)$ to the point $[\alpha'(\mu), \beta'(\mu)]$ which lies on the curve C_s.

The functions $|\Omega^{\pm}(\mu,s)/c|^2$ are easily found from Eq. (4) to be

$$\left|\frac{\Omega^{\pm}(\mu,s)}{c}\right|^2 = (\alpha - \mu \tanh^{-1} \mu)^2 + (\beta \pm \pi\mu/2)^2 \quad , \qquad (5)$$

while the parametric form of Eq. (2) is

$$|\beta'(\mu)| = \pi\mu/2 \quad , \quad \alpha'(\mu) = \mu \tanh^{-1} \mu \quad , \quad 0 \leq \mu \leq 1 \quad . \qquad (6)$$

Thus $|\Omega^{\pm}(\mu,s)/c|^2$ are the squares of the distances in the s/c-plane from the point (α,β) to the points $[\alpha'(\mu), \mp|\beta'(\mu)|]$ which lie on the curve C_s, in the lower and upper half-planes, respectively. When $\beta = 0$, $\Omega^+(\mu,s)$ and $\Omega^-(\mu,s)$ are complex conjugates so that Eq. (5) reduces to Eq. (3) and the stated result follows.

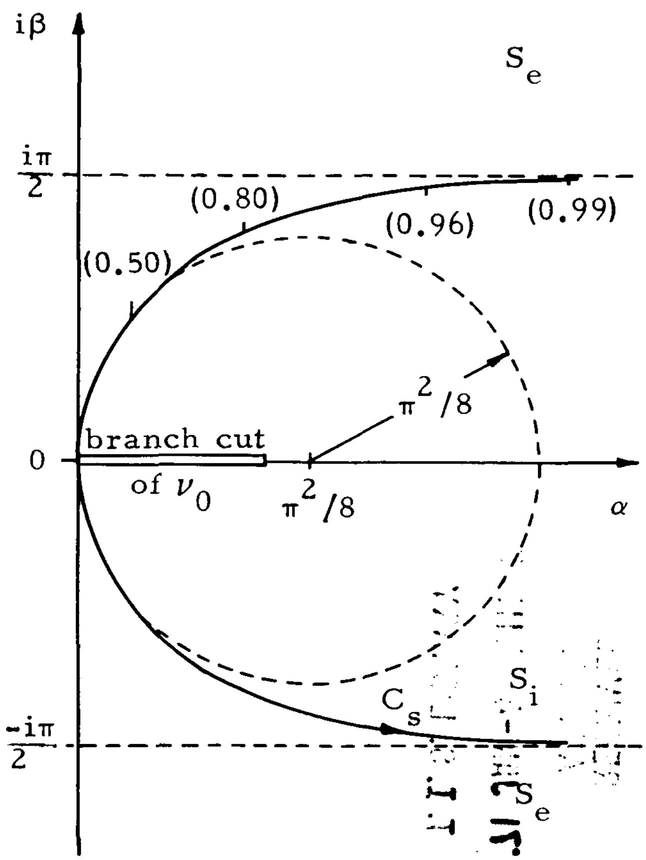

Fig. 1. Location of the curve C_s in the complex s/c-plane, $s/c = \alpha + i\beta$. Several values of the parameter μ are given in parentheses for C_s in the upper half-plane. The function $\nu_0(s)$ is the solution of $\Omega(\nu_0, s) = 0$ for which $\mathrm{Re}(\nu_0) > 0$ when $\mathrm{Re}(s/c) > 1$. Some values of $\nu_0(s)$ are plotted in the complex s-plane in Fig. 1 of Ref. 5 (Fig. 7.1 of Ref. 2).

Case et al. show, for example, that $g(1/\alpha,\mu)|_{\max}$ occurs at $\mu = 0$ for $\alpha < \pi^2/8$, whereas for $\alpha > \pi^2/8$ it occurs for μ between 0 and 1. For α very large, they show that $g_{\max} \to 4\alpha^2/\pi^2$. The present geometric interpretation is seen to be consistent with these characteristics. The radius of curvature of the curve C_s given by Eq. (2) is $\pi^2/8$ at $(\alpha',\beta') = (0,0)$ as indicated by the dashed circle on Fig. 1. For α' very large, $\beta' \to \pi/2$ so that the minimum squared distance from $(\alpha,0)$ to (α',β') approaches $\pi^2/4$, in agreement with Eq. (3) and $g_{\max} \to 4\alpha^2/\pi^2$.

[1] K. M. CASE, F. deHOFFMAN, and G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, U.S. Government Printing Office, Washington, D.C. (1953).

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page - 1
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